Nonperturbative contributions in an analytic running coupling of QCD

Aleksey I. Alekseev [1]
Institute for High Energy Physics, 142280 Protvino, Moscow Region, Russia

Abstract

In the framework of analytic approach to QCD the nonperturbative contributions in running coupling of strong interaction up to 4-loop order are obtained in an explicit form. For all $Q > \Lambda$ they are shown to be represented in the form of an expansion in inverse powers of Euclidean momentum squared. The expansion coefficients are calculated for different numbers of active quark flavors n_f and for different number of loops taken into account. On basis of the stated expansion the effective method for precise calculation of the analytic running coupling can be developed.

PACS number(s): 12.38.Aw, 12.38.Lg

It is widely believed that nonphysical singularities of the perturbation theory in the infrared region of QCD should be canceled by the nonperturbative contributions. The nonperturbative contributions arise quite naturally in an analytic approach [1] — [5] to QCD. The idea of the approach goes back to Refs. [6, 7] devoted to the nonphysical ghost pole problem in QED. In recent papers [2, 3] it is suggested to solve the ghost pole problem in QCD demanding the running coupling constant be analytic in Q^2 (Q^2 is the Euclidean momentum squared). As a result of the procedure, instead of the one-loop expression $\alpha_s^{(1)}(Q^2) = (4\pi/b_0)/\ln(Q^2/\Lambda^2)$ taking into account the leading logarithms and having the ghost pole at $Q^2 = \Lambda^2$, one obtains new expression

$$\alpha_{an}^{(1)}(Q^2) = \frac{4\pi}{b_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right]. \tag{1}$$

Eq. (1) is an analytic function in the complex Q^2-plane with a cut along the negative real semiaxis. The pole of the perturbative running coupling at $Q^2 = \Lambda^2$ is canceled by the nonperturbative contribution ($\Lambda^2 \simeq \mu^2 \exp\{-4\pi/(b_0\alpha_{an}(\mu^2))\}$ at $\alpha_{an}(\mu^2) \to 0$) and the value $\alpha_{an}^{(1)}(0) = 4\pi/b_0$ appeared to be finite and independent of the normalization conditions (independent of Λ). The most important feature of the procedure is the stability property [2, 3] of the value of the "analytically improved" running coupling constant at

[1] Electronic address: alekseev@mx.ihep.su

zero with respect to high corrections, $\alpha_{an}^{(1)}(0) = \alpha_{an}^{(2)}(0) = \alpha_{an}^{(3)}(0)$. This property provides the high corrections stability of $\alpha_{an}(Q^2)$ in the whole infrared region. We shall demonstrate that $\alpha_{an}(0) = 4\pi/b_0$ for any finite loop order initial $\alpha_s(Q^2)$ in the standard inverse powers of logarithms expansion form being a consequence of the asymptotic freedom property.

Note that for the one-loop case we have not only the spectral representation for $\alpha_{an}(Q^2)$ but, in the first place, the nonperturbative contribution is extracted from the analytic running coupling explicitly,

$$\alpha_{an}(Q^2) = \alpha^{pt}(Q^2) + \alpha_{an}^{npt}(Q^2), \tag{2}$$

and, in the second place, the 1-loop order nonperturbative contribution can be presented as convergent at $Q^2 > \Lambda^2$ of constant signs series in the inverse powers of the momentum squared,

$$\alpha_{an}^{npt}(Q^2) = \frac{4\pi}{b_0} \sum_{n=1}^{\infty} c_n \left(\frac{\Lambda^2}{Q^2}\right)^n, \tag{3}$$

where $c_n = -1$. For the standard as well as for the iterative 2-loop perturbative input the nonperturbative contributions in the analytic running coupling are calculated explicitly in Ref. [8]. In the ultraviolet region the nonperturbative contributions also can be represented as a series in inverse powers of the momentum squared. The 3-loop case is considered in Ref. [9] where, to handle the singularities originating from the perturbative input, the method which is more general then that of Ref. [8] was developed.

In this paper we extract in an explicit form the nonperturbative contributions to $\alpha_{an}(Q^2)$ up to 4-loop order in analytic approach to QCD, and find the coefficients c_n of the expansion of the form (3). This gives the effective method for a precise calculation of $\alpha_{an}(Q^2)$ which is not connected with numerical integration.

The behavior of the QCD running coupling $\alpha_s(Q^2)$ is defined by the renormalization group equation

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s) = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \beta_2 \alpha_s^4 + \beta_3 \alpha_s^5 + O(\alpha_s^6), \tag{4}$$

where the coefficients [10] — [13]

$$\begin{aligned}
\beta_0 = & - \frac{1}{4\pi} b_0, \quad b_0 = 11 - \frac{2}{3} n_f, \\
\beta_1 = & - \frac{1}{8\pi^2} b_1, \quad b_1 = 51 - \frac{19}{3} n_f, \\
\beta_2 = & - \frac{1}{128\pi^3} b_2, \quad b_2 = 2857 - \frac{5033}{9} n_f + \frac{325}{27} n_f^2, \\
\beta_3 = & - \frac{1}{256\pi^4} b_3, \quad b_3 = \frac{149753}{6} + 3564\zeta_3 \\
& - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right) n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) n_f^2 + \frac{1093}{729} n_f^3.
\end{aligned} \tag{5}$$

Here n_f is number of active quark flavors and ζ is Riemann zeta-function, $\zeta_3 = \zeta(3) = 1.202056903\ldots$. The first two coefficients β_0, β_1 do not depend on the renormalization

scheme choice. The next coefficients do depend on this choice. Being calculated within the $\overline{MS}$-scheme in an arbitrary covariant gauge for the gluon field they appeared to be independent of the gauge parameter choice. Integrating Eq. (4) yields

$$\begin{aligned} \frac{1}{\alpha_s(Q^2)} &+ \frac{\beta_1}{\beta_0}\ln\alpha_s(Q^2)+\frac{1}{\beta_0^2}\left(\beta_0\beta_2-\beta_1^2\right)\alpha_s(Q^2)+\frac{1}{2\beta_0^3}\left(\beta_1^3-2\beta_0\beta_1\beta_2+\beta_0^2\beta_3\right)\alpha_s^2(Q^2) \\ &+ O(\alpha_s^3(Q^2))=-\beta_0\ln(Q^2/\Lambda^2)+\tilde{C}. \end{aligned} \tag{6}$$

The integration constant is represented here as a combination of two constants Λ and $\tilde{C}$. Dimensional constant Λ is a parameter which defines the scale of Q and is used for developing the iteration procedure. Iteratively solving Eq. (6) for $\alpha_s(Q^2)$ at $L=\ln(Q^2/\Lambda^2)\to\infty$ one obtains

$$\begin{aligned} \alpha_s(Q^2)=-\frac{1}{\beta_0 L}\Bigg\{1+\frac{\beta_1}{\beta_0^2 L}(\ln L+C)+\frac{\beta_1^2}{\beta_0^4 L^2}\left[(\ln L+C)^2-(\ln L+C)-1+\frac{\beta_0\beta_2}{\beta_1^2}\right] \\ +\frac{\beta_1^3}{\beta_0^6 L^3}\left[(\ln L+C)^3-\frac{5}{2}(\ln L+C)^2-\left(2-\frac{3\beta_0\beta_2}{\beta_1^2}\right)(\ln L+C)+\frac{1}{2}-\frac{\beta_0^2\beta_3}{2\beta_1^3}\right]+O\left(\frac{1}{L^4}\right)\Bigg\}, \end{aligned} \tag{7}$$

where $C=\ln(-\beta_0)+(\beta_0/\beta_1)\tilde{C}$. Within the conventional definition of Λ as $\Lambda_{\overline{MS}}$ [14] one chooses $C=0$. At that the functional form of the approximate solution for $\alpha_s(Q^2)$ turns out to be somewhat simpler, but it requires distinct $\Lambda_{\overline{MS}}$ for different n_f. With this choice Eq. (7) at three loop level corresponds to the standard solution written in the form of the expansion in inverse powers of logarithms [15], and at four loop level it corresponds to [16]. We shall deal with nonzero C because this freedom can be useful for an optimization of the finite order perturbation calculations. Moreover, in the presence of the n_f-dependent constant C it is possible to construct matched solution of Eq. (4) with universal n_f independent constant Λ [17].

Let us introduce the function $\Phi(z)$ of the form

$$\begin{aligned} \Phi(z) &= \frac{1}{z}-b\frac{\ln(z)+C}{z^2}+b^2\left[\frac{(\ln(z)+C)^2}{z^3}-\frac{\ln(z)+C}{z^3}+\frac{\kappa}{z^3}\right] \\ &- b^3\left[\frac{(\ln(z)+C)^3}{z^4}-\frac{5}{2}\frac{(\ln(z)+C)^2}{z^4}+(3\kappa+1)\frac{\ln(z)+C}{z^4}+\frac{\bar{\kappa}}{z^4}\right], \end{aligned} \tag{8}$$

where the coefficients b, κ, and $\bar{\kappa}$ are equal to

$$\begin{aligned} b &= -\frac{\beta_1}{\beta_0^2}=\frac{2b_1}{b_0^2}, \\ \kappa &= -1+\frac{\beta_0\beta_2}{\beta_1^2}=-1+\frac{b_0b_2}{8b_1^2}, \\ \bar{\kappa} &= \frac{1}{2}-\frac{\beta_0^2\beta_3}{2\beta_1^3}=\frac{1}{2}-\frac{b_0^2b_3}{16b_1^3}. \end{aligned} \tag{9}$$

To choose the main branch of the multivalued function (8) we cut complex z-plane along the negative semiaxis. Then the solution (7) can be written as $\alpha_s(Q^2)=(4\pi/b_0)a(x)$,

where $a(x) = \Phi(\ln x)$, $x = Q^2/\Lambda^2$. Function $a(x)$ is unambiguously defined in the complex x-plain with two cuts along the real axis, physical cut from minus infinity to zero and nonphysical one from zero to unity. At $x \simeq 1$ the perturbative running coupling has singularities of a different analytical structure. Namely, at $x \simeq 1$ the leading singularities are

$$\begin{aligned} a^{(1)}(x) &\simeq \frac{1}{x-1}, \;\; a^{(2)}(x) \simeq -\frac{b}{(x-1)^2}\ln(x-1), \\ a^{(3)}(x) &\simeq \frac{b^2}{(x-1)^3}\ln^2(x-1), \;\; a^{(4)}(x) \simeq -\frac{b^3}{(x-1)^4}\ln^3(x-1). \end{aligned} \tag{10}$$

This is not an obstacle for the analytic approach which removes all this nonphysical singularities.

The analytic running coupling is defined by the spectral representation

$$a_{an}(x) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{x+\sigma}\rho(\sigma), \tag{11}$$

with the spectral density $\rho(\sigma) = \operatorname{Im} a_{an}(-\sigma - i0) = \operatorname{Im} a(-\sigma - i0)$ where $a(x)$ is the perturbative running coupling. It is seen that dispersively modified coupling of the form (11) has analytical structure which is consistent with causality.

By making the analytic continuation of Eq. (8) into the Minkowski space by formal substitution $x = -\sigma - i0$ one obtains the spectral density as an imaginary part of $\Phi(\ln\sigma - i\pi)$. Function $a(x)$ is regular and real for real $x > 1$. So, to find the spectral density $\rho(\sigma)$ we can use the reflection principle $(a(x))^* = a(x^*)$ where x is considered as a complex variable. Then

$$\rho(\sigma) = \frac{1}{2i}\left(\Phi(\ln\sigma - i\pi) - \Phi(\ln\sigma + i\pi)\right). \tag{12}$$

By the change of variable of the form $\sigma = \exp(t)$, the analytical expression is derived from (11), (12) as follows:

$$a_{an}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \frac{e^t}{x+e^t} \times \left\{\Phi(t - i\pi) - \Phi(t + i\pi)\right\}. \tag{13}$$

Let us prove that $a_{an}(0) = 1$. It follows from Eq. (13) that

$$a_{an}(0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \left\{\Phi(t - i\pi) - \Phi(t + i\pi)\right\}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \left\{\left[\Phi(t - i\pi) - \frac{1}{t - i\pi}\right] - \left[\Phi(t + i\pi) - \frac{1}{t + i\pi}\right] + \left[\frac{1}{t - i\pi} - \frac{1}{t + i\pi}\right]\right\}. \tag{14}$$

For the first term in Eq. (14) close the integration contour in the lower half-plane of the complex variable t by the arch of the "infinite" radius without affecting the value of the integral. We can do it because the integrand multiplied by t goes to zero at $\mid t \mid \to \infty$. There are no singularities inside the contour, so we obtain zero contribution from the term

considered. For the second term we close the integration contour in the upper half-plane of the complex variable t with the same result. So we have

$$a_{an}(0) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} dt \left[\frac{1}{t - i\pi} - \frac{1}{t + i\pi} \right] = 1. \tag{15}$$

For any finite loop order the expansion structure of the perturbative solution in inverse powers of logarithms ensure the property of the analytical coupling $a_{an}(0) = 1$. The arguments are suitable for all solutions $\Phi(z)$ if singularities are situated at the real axis of complex z-plane in particular for the iterative solutions of Refs. [2, 3].

Let us see what the singularities of the integrand of (13) in the complex t-plane are. First of all the integrand has simple poles at $t = \ln x \pm i\pi(1 + 2n)$, $n = 0, 1, 2, \ldots$. All the residues of the function $\exp(t)/(x + \exp(t))$ at these points are equal to unity. Apart from this poles the integrand of (13) has at $t = \pm i\pi$ poles up to fourth order and logarithmic type branch points which coincide with poles from second order to fourth order. Let us cut the complex t-plane in a standard way, $t = \pm i\pi - \lambda$, with λ being the real parameter varying from 0 to ∞. Append the integration by the arch of the "infinite" radius without affecting the value of the integral. Close the integration contour C_1 in the upper half-plane of the complex variable t excluding the singularities at $t = i\pi$. In this case an additional contribution emerges due to the integration along the sides of the cut and around the singularities at $t = i\pi$. The corresponding contour we denote as C_2.

Let us turn to the integration along the contour C_1. For the integrand of Eq. (13) which we denote as $F(t)$ the residues at $t = \ln x + i\pi(1 + 2n)$, $n = 0, 1, 2, \ldots$ are as follows

$$\mathrm{Res}\, F(t) \mid_{t=\ln x + i\pi(1+2n)} = \Phi(\ln x + 2\pi i n) - \Phi(\ln x + 2\pi i(n + 1)). \tag{16}$$

By using the residue theorem one readily obtains the contribution $\Delta(x)$ to the integral (13) from the integration along contour C_1. It reads

$$\Delta(x) = \frac{1}{2\pi i} \int\limits_{C_1} F(t)\, dt = \sum_{n=0}^{\infty} \mathrm{Res}\, F\left(t = \ln x + i\pi(1 + 2n)\right) = \Phi(\ln x). \tag{17}$$

We can see that this contribution is exactly equal to the initial $a(x)$. Therefore we call it a perturbative part of $a_{an}(x)$, $a^{pt}(x) = \Delta(x)$. The remaining contribution of the integral along the contour C_2 can naturally be called a nonperturbative part of $a_{an}(x)$ according to Eq. (2). Let us turn to the calculation of $a^{npt}_{an}(x)$. We have

$$a^{npt}_{an}(x) = \frac{1}{2\pi i} \int\limits_{C_2} dt\, \frac{e^t}{x + e^t} \times \left\{ \Phi(t - i\pi) - \Phi(t + i\pi) \right\}. \tag{18}$$

We consider x as real variable, $x > 1$. One can omit the term of the integrand in Eq. (18) which has singularities at $t = -i\pi$. Let us change the variable $t = z + i\pi$ and introduce the function

$$f(z) = \frac{1}{1 - x \exp(-z)}. \tag{19}$$

Then we can rewrite Eq. (18) in the form

$$a_{an}^{npt}(x) = \frac{1}{2\pi i}\int\limits_{C} dz\, f(z)\left\{\frac{1}{z} - b\left[\frac{\ln(z)}{z^2} + \frac{C}{z^2}\right] + b^2\left[\frac{\ln^2(z)}{z^3} + (2C-1)\,\frac{\ln z}{z^3} + \frac{\kappa - C + C^2}{z^3}\right]\right.$$

$$\left. - b^3\left[\frac{\ln^3(z)}{z^4} + \left(3C - \frac{5}{2}\right)\frac{\ln^2}{z^4} + (3C^2 - 5C + 3\kappa + 1)\frac{\ln z}{z^4} + \frac{C^3 - \frac{5}{2}C^2 + (3\kappa+1)C + \bar{\kappa}}{z^4}\right]\right\}. \tag{20}$$

The cut in the complex z-plane goes now from zero to $-\infty$. Starting from $z = -\infty - i0$ the contour C goes along the lower side of the cut then goes around the origin, and then it goes further along the upper side of the cut to $z = -\infty + i0$. The contour C can be chosen in such a way that it does not envelop a "superfluous" singularities corresponding to the perturbative contributions. Using the technique of Ref. [9] one can integrate the terms of Eq. (20) and obtain

$$a_{an}^{npt}(x) = -\frac{1}{x-1} + b\left\{\frac{(1+C)x}{(x-1)^2} + x\int\limits_0^1 d\sigma\, \ln(-\ln\sigma)\frac{x+\sigma}{(x-\sigma)^3}\right\}$$

$$-\frac{1}{2}b^2\left\{\left[1 - \frac{\pi^2}{3} + \kappa + (1+C)^2\right]\frac{x(x+1)}{(x-1)^3} + x\int\limits_0^1 d\sigma\left[2(1+C)\ln(-\ln\sigma) + \ln^2(-\ln\sigma)\right]\right.$$

$$\left.\times\frac{x^2 + 4x\sigma + \sigma^2}{(x-\sigma)^4}\right\} + \frac{1}{6}b^3\left\{\left[2 + \frac{5}{2}\kappa + \bar{\kappa} + 3(1+C)\left(1 - \frac{\pi^2}{3} + \kappa\right) + (1+C)^3\right]\right.$$

$$\times\frac{x(x^2+4x+1)}{(x-1)^4} + x\int\limits_0^1 d\sigma\left[3\left(1 - \frac{\pi^2}{3} + \kappa + (1+C)^2\right)\right.$$

$$\left.\left.\times\ln(-\ln\sigma) + 3(1+C)\ln^2(-\ln\sigma) + \ln^3(-\ln\sigma)\right]\frac{x^3 + 11x^2\sigma + 11x\sigma^2 + \sigma^3}{(x-\sigma)^5}\right\}. \tag{21}$$

This formula gives the nonperturbative contributions in the explicit form. Expanding Eq. (21) in inverse powers of x we have

$$a_{an}^{npt}(x) = \sum_{n=1}^{\infty}\frac{c_n}{x^n}. \tag{22}$$

Making the change of variable $\sigma = \exp(-t)$ and integrating [18], [19] over t we finally find

$$c_n = -1 + bn\left[1 + C - \gamma - \ln(n)\right] - \frac{1}{2}b^2n^2\left[1 - \frac{\pi^2}{6} + \kappa + \Big(1 + C - \gamma - \ln(n)\Big)^2\right]$$

$$+\frac{1}{6}b^3n^3\left[2 + \frac{5}{2}\kappa + \bar{\kappa} - 2\zeta_3 + \Big(1 + C - \gamma - \ln(n)\Big)^3 + 3\Big(1 + C - \gamma - \ln(n)\Big)\left(1 - \frac{\pi^2}{6} + \kappa\right)\right]. \tag{23}$$

Here γ is the Euler constant, $\gamma \simeq 0.5772$. We can see from Eq. (23) that power series (22) is uniformly convergent at $x > 1$ and its convergence radius is equal to unity. For numerical

Table 1: The dependence of c_1 on n_f for 1-loop, 2-loop, 3-loop, and 4-loop cases.

n_f	c_1^{1-loop}	Δ_{2-loop}	Δ_{3-loop}	Δ_{4-loop}	c_1^{2-loop}	c_1^{3-loop}	c_1^{4-loop}
0	-1.0	0.35640	-0.01568	-0.03900	-0.64360	-0.65929	-0.69828
3	-1.0	0.33405	0.01608	-0.07825	-0.66595	-0.64987	-0.72812
4	-1.0	0.31252	0.04949	-0.11006	-0.68748	-0.63799	-0.74805
5	-1.0	0.27813	0.11653	-0.16002	-0.72187	-0.60535	-0.76537
6	-1.0	0.22433	0.25378	-0.22731	-0.77567	-0.52189	-0.74920

evaluation of the coefficients c_n assume that $C = 0$. Then the coefficients c_n depend on n, n_f and number of loops taken into account. The 1-loop order contribution to c_n equals to -1 for all n and n_f. Up to 4-loop approximation the coefficients c_n for all n, n_f are negative. With the exception of the 3-loop case at $n_f = 6$, the 2 — 4-loop coefficients c_n for $n_f = 0, 3, 4, 5, 6$ monotonously increases in the absolute value with increasing n. In the ultraviolet region ($x \gg 1$) the nonperturbative contributions are determined by the first term of the series (22). In Table 1 we give the values of c_1 and loop corrections for $n_f = 0, 3, 4, 5, 6$. One can see that for all n_f up to four loops c_1 is of the order of unity. An account for the high loop corrections results in some compensation of the 1-loop leading at large x term of the form $1/x$. The relative error of the approximation of the analytic running coupling with only one first term of the series (22) for the nonperturbative contributions taken into account,

$$\alpha_{an}(Q^2) \simeq \alpha^{pt}(Q^2) - \frac{4\pi}{b_0}\left\{1 - b\,(1-\gamma) + \frac{1}{2}b^2\left(1 - \frac{\pi^2}{6} + \kappa + (1-\gamma)^2\right)\right.$$
$$\left. -\frac{1}{6}b^3\left[2 + \frac{2}{5}\kappa + \bar{\kappa} - 2\zeta_3 + (1-\gamma)^3 + 3\,(1-\gamma)\left(1 - \frac{\pi^2}{6} + \kappa\right)\right]\right\}\frac{\Lambda^2}{Q^2}, \tag{24}$$

is shown in Fig. 1 for 1 — 4-loop cases. We have chosen $n_f = 4$ within the region of Q considered.

As a result of the expansion coefficients increase not too fast, the representation of the analytic running coupling of QCD in the form (2), (3) with c_n as (23) provide one with the effective method for the calculation of α_{an} at $Q > \Lambda$. At that there is no need for the summation of large number of terms of the series. Approximation of the nonperturbative "tail" by the leading term already at $Q \sim 5\Lambda$ gives one percent accuracy for α_{an}. As regards the parameters $\Lambda^{(n_f=5)}$, $\Lambda^{(n_f=4)}$ of the analytic approach they turn out to be close to the perturbative values as a consequence of the rapid decrease of the nonperturbative contributions in the ultraviolet region.

I am indebted to B.A. Arbuzov, V.A. Petrov, V.E. Rochev for useful discussions. This work was supported in part by the Russian Foundation for Basic Research under grant No. 99-01-00091.

References

[1] I.L. Solovtsov and D.V. Shirkov, Theor. Math. Phys. 120 (1999) 1210.

[2] D.V. Shirkov and I.L. Solovtsov, JINR Rapid Comm. 2[76]-96 (1996) 5.

[3] D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett. 79 (1997) 1209.

[4] Yu.L. Dokshitzer, G. Marchesini and B.R. Webber, Nucl. Phys. B 469 (1996) 93.

[5] G. Grunberg, hep-ph/9705290, hep-ph/9711481, hep-ph/9807494.

[6] P.J. Redmond, Phys. Rev. 112 (1958) 1404.

[7] N.N. Bogolubov, A.A. Logunov and D.V. Shirkov, Sov. Phys. JETP 37 (1959) 805.

[8] A.I. Alekseev, Phys. Rev. D 61 (2000) 114005.

[9] A.I. Alekseev, IHEP preprint 2000-40, Protvino, 2000; hep-ph/0011178.

[10] D.J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343; H.D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.

[11] D.R.T. Jones, Nucl. Phys. B 75 (1974) 531; W.E. Caswell, Phys. Rev. Lett. 33 (1974) 244; E.Sh. Egoryan and O.V. Tarasov, Teor. Mat. Fiz. 41 (1979) 26 [Theor. Math. Phys. 41 (1979) 863].

[12] O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, Phys. Lett. B 93 (1980) 429; S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B 303 (1993) 334.

[13] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B 400 (1997) 379; J. Ellis, M. Karliner and M.A. Samuel, Phys. Lett. B 400 (1997) 176.

[14] W.A. Bardeen, A. Buras, D. Duke and T. Muta, Phys. Rev. D 18 (1978) 3998.

[15] Particle Data Group, D.E. Groom et al., Eur. Phys. J. C 15 (2000) 85.

[16] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, Phys. Rev. Lett. 79 (1997) 2184.

[17] W.J. Marciano, Phys. Rev. D 29 (1984) 580.

[18] H. Bateman and A. Erdelyi, Higher transcendental functions, Vol. 1 (McGraw-Hill, 1953).

[19] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, Integrals and Series, Vol. 1 (Gordon and Breach, 1986).

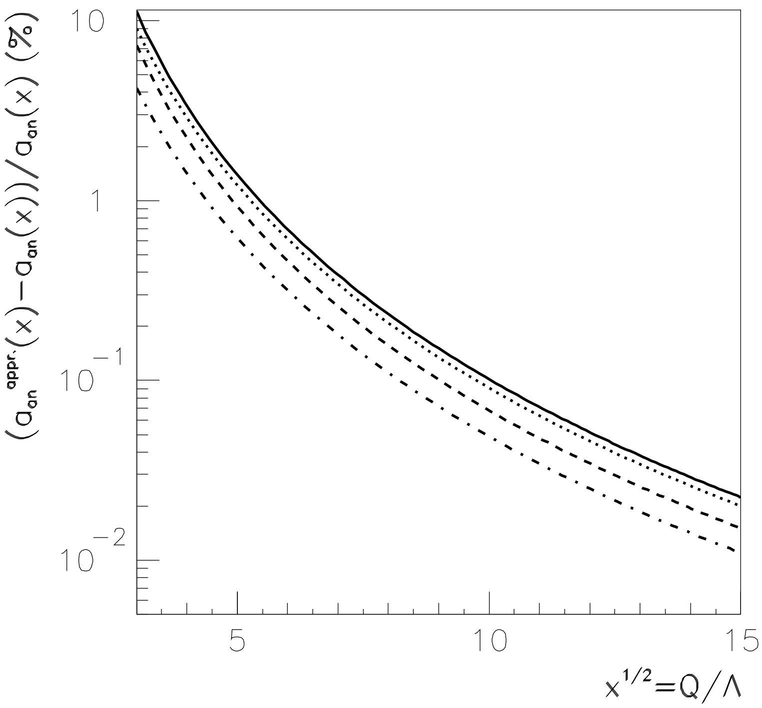

Figure 1: Relative error of the approximation of a_{an} with a_{an}^{npt} approximated by one first term of the series, as function of $x^{1/2} = Q/\Lambda$ for 1 — 4-loop cases at $n_f = 4$. Dash-dotted line, dotted line, dashed line and solid line correspond to 1-loop, 2-loop, 3-loop and 4-loop cases respectively.